J KAU: Eng. Sci., **vol. 3**, pp. 103-111 (1411 A.H./1991 A.D.)

A Technique to Eliminate the Bounds in Nonlinear Programming Problems

ABHA GOEL and G. C. SHARMA
Department of Mathematics, Institute of Basic Science, Agra University, Agra, India.

ABSTRACT In this paper, we consider the problem of minimization of an objective function having continuous first and second partial derivatives, subject to nonnegativity restrictions or upper and lower bounds on the variables of a nonlinear programming problem. An appropriate transformation of its variables has been made to convert such a constrained optimization problem into an unconstrained optimization problem. Some conditions are developed to guarantee that every local minimum of the unconstrained problem satisfies the Kuhn-Tucker's first order necessary conditions for a local minimum of the constrained problem. There are some conditions for which the transformed objective function maintains the convexity of the original objective function in a neighbourhood of the solution.

KEY WORDS : Transformation of variables, Bounds, Nonlinear programming problem, Convexity.

1. Introduction

The transformation techniques help in solving a nonlinear programming problem by reducing it to one or more unconstrained programs. Such a reduction is brought about by using a transformation that combines the objective function and the constraints. The advantage in using a transformation technique is that, problems without constraints are easier to be solved than those with constraints, because many techniques are available to solve the unconstrained programs.

In this paper, we consider a given nonlinear programming problem.

$$\begin{aligned} &\text{Minimize} \quad f(y_1, y_2, \ldots, y_n). \\ &\text{Subject to} \quad y_j \geq 0 \quad , \quad j = 1, 2, \ldots, m. \\ &\qquad\qquad l_j \leq y_j \leq u_j \quad , \quad j = m+1, \ldots, n \end{aligned} \tag{1}$$

where f is a nonlinear function having continuous first and second partial derivatives.

In 1966, Box[1] presented some transformations that might be used to combine the objective function and the constraints.

Here, we take the transformation

$$y_j = T_j(x_j), \quad j = 1, 2, \ldots, n \tag{2}$$

Then the unconstrained problem is to

$$\text{Minimize } F(x_1, x_2, \ldots, x_n) = f[T_1(x_1), T_2(x_2), \ldots., T_n(x_n)]. \tag{3}$$

Now we have to solve problem (3), instead of solving the problem (1).

In section 2, we give some conditions that a transformation must satisfy in order, that it does not have any additional local minima for the unconstrained problem. In section 3, we will give a criterion, *i.e.*, to what extent, the transformation described in section 2 maintains convexity.

2. Selection of an Appropriate Transformation

Some conditions that the transformation (2) should satisfy are suggested by Powell[2], which we call the global mapping conditions

(a) If x_j is any real number, then $T_j(x_j)$ should satisfy the constraints on y_j.

(b) There should exist a value of x_j, for every feasible value of y_j such that $y_j = T_j(x_j)$.

Following two cases are likely to occure:

Case 1

In some cases the global mapping conditions prevent the introduction of additional local minima into the problem. For Example

Example 1

Minimize $f(y_1, y_2) = (y_1 - \alpha)^2 + (y_2 - \beta)^2$,

Subject to $y_j \geqslant 0$, $j = 1, 2$.

Solution is $y^* = (\alpha, \beta)^T$, $f(y^*) = 0$.

Let $y_j = T_j(x_j) = (x_j^2 - 1)^2 \geqslant 0, \quad j = 1, 2$.

Global mapping conditions are satisfied by this tansformation and the transformed problem is :

$$\text{Minimize } F(x_1, x_2) = (x_1^4 - 2x_1^2 + 1 - \alpha)^2 + (x_2^4 - 2x_2^2 + 1 - \beta)^2$$

which has four local minima and all of these map into $(\alpha, \beta)^T$, the solution of the original problem if $\alpha, \beta \leqslant 1$. Thus in this cae the global mapping conditions are sufficient in order that a transformation does not have any additional local minima for the unconstrained problem.

Case 2

In some cases the global mapping conditions do not prevent the introduction of additional local minima into the problem. For example

Example 2

Minimize $f(y_1, y_2) = (y_1 - 4)^2 + (y_2 - 9)^2$

Subject to $y_j \geqslant 0 \quad , \quad j = 1, 2.$

Solution is $y^* = (4, 9)^T \ , \ f(y^*) = 0.$

Let $y_j = T_j(x_j) = (x_j^2 - 1)^2 \geqslant 0, \quad j = 1, 2.$

Global mapping conditions are satisfied by this tansformation. The transformed problem is :

Minimize $F(x_1, x_2) = (x_1^4 - 2x_1^2 - 3)^2 + (x_2^4 - 2x_2^2 - 8)^2$

which has nine local minima. Four of these map into $(4, 9)^T$, the solution of the original problem, but remaining five $(0, 0)$, $(0, \pm 2)$, $(\pm \sqrt{3}, 0)$ do not map into local minimum of f.

Here we see that there are some additional local minima in the solution of the transformed unconstrained problem. We require that any local minimum of the transformed problem (3) should satisfy the first order necessary conditions (Kuhn-Tucker's) for a local minimum of the original problem (1). We find the following conditions on T_j, $j = 1, 2, \ldots, n$, are sufficient to this purpose which we call the local mapping conditions

(a) T_j are functions with continuous first and second partial derivatives, where $j = 1, 2, \ldots, n$.

(b) $T_j(x_j) \geqslant 0$ for $j = 1, 2, \ldots, m$.
Also, $T'_j(x_j) = 0 \Rightarrow T_j(x_j) = 0$ and $T''_j(x_j) > 0$.

(c) $l_j \leqslant T_j(x_j) \leqslant u_j$ for $j = m + 1, \ldots, n$.
Also, $T'_j(x_j) = 0 \Rightarrow$ either $T_j(x_j) = l_j$ and $T''_j(x_j) > 0$ or $T_j(x_j) = u_j$ and $T''_j(x_j) < 0$.

The following theorem is the required faithfulness of the mapping.

Theorem

Suppose T_j, $j = 1, 2, \ldots, n$, satisfy the local mapping conditions. Let x be a stationary point of F for which $\nabla^2 F(x)$ is positive semidefinite. Then, for a local minimum of the original problem (1), $y = T(x)$ satisfies the first order necessary conditions, where $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ are the point in n-dimensional Euclidean space and $T(x) = [T_1(x_1), T_2(x_2), \ldots, T_n(x_n)]^T$.

Proof

$$\text{Since } \nabla F(x) = [f_1' T_1'(x_1), f_2' T_2'(x_2), \ldots, f_n' T_n'(x_n)]^T$$

where f_j' denotes the first partial derivative of f with respect to y_j.

If x is stationary point of F, then for each value of j, where $j = 1,2,..,n$.

$$\text{either } \quad f_j' = 0 \quad \text{or} \quad T_j'(x_j) = 0 \tag{4}$$

By the local mapping conditions, if $T_j'(x) = 0$ then $y_j = T_j(x_j)$ lies on a boundary of the constraints of (1). It can be shown that the first order necessary conditions for a local minimum of the original problem (1) will be satisfied by a stationary point F if and only if

$$\left.\begin{array}{ll} & f_j' \, T(x) \geqslant 0 \quad \text{for} \quad j = 1, 2, \ldots, m \\ \text{and} & \text{if, for} \quad j = m+1, \ldots, n, \\ & f_j' \, T(x) \geqslant 0 \quad \text{when} \quad y_j = l_j \\ \text{and} & f_j' \, T(x) \geqslant 0 \quad \text{when} \quad y_j = u_j \end{array}\right\} \tag{5}$$

By assumption, we are at a stationary point of F at which the Hessian of F is positive semidefinite. We will now show that this automatically guarantees that the conditions (5) on f_j' hold.

It can be seen that

$$\begin{array}{ll} & \nabla^2 F(x) = D \nabla^2 f(y) D + E \\ \text{where} & D = \operatorname{diag}\{T_j'(x_j) : j = 1,2,\ldots,n\}, \\ \text{and} & E = \operatorname{diag}\{f_j' T_j''(x_j) : j = 1,2,\ldots,n\} \end{array}$$

At a stationary point x, if $\nabla^2 F(x)$ is positive semidefinite, then $f_j' T_j''(x_j) \geqslant 0$ for any value of j for which $T_j'(x_j) = 0$.

$$\begin{array}{ll} & \text{By (4), if } T_j'(x_j) \neq 0, \quad \text{then } f_j' = 0 \quad \text{for} \quad j = 1,2,\ldots,n. \\ & \text{If} \quad T_j'(x_j) = 0, \quad \text{then by the local mapping conditions,} \\ & f_j' T_j''(x_j) \geqslant 0 \Rightarrow f_j' \geqslant 0 \quad \text{for} \quad j = 1,2,\ldots,m, \\ \text{and} & f_j' T_j''(x_j) \geqslant 0 \Rightarrow \text{ either } y_j = l_j \text{ and } f_j' \geqslant 0 \\ & \text{or} \quad y_j = u_j \text{ and } f_j' \geqslant 0 \quad \text{for} \quad j = m+1,\ldots,n. \end{array}$$

These are the first order necessary conditions.

The local mapping conditions guarantee that any local minimum of (3) satisfies the first order necessary conditions for a local minimum of (1) but these conditions have to guarantee that a solution of (3) can actually be found. For example

Example 3

$$\text{Minimize} \qquad f(y_1, y_2) = y_1 + 2y_2$$

Subject to $\quad -1 \leq y_j \leq 1, \quad j = 1,2$

The solution $\quad y^* = (-1, -1)^T$ and $f(y^*) = -3$

Let $y_j = T_j(x_j) = \dfrac{2x_j}{x_j^2 + 1}, \; j = 1,2$

Local and global mapping conditions are satisfied by this transformation and the transformed problem is

$$\text{Minimize} \quad F(x_1, x_2) = \frac{2x_1}{x_1^2 + 1} + \frac{4x_2}{x_2^2 + 1}$$

We use steepest descent method (see Flecther[3] or Powell[2]) to solve the transformed problem,

$$\nabla F(x_1, x_2) = \left[\frac{2 - 2x_1^2}{(x_l^2 + 1)^2}, \frac{4 - 4x_2^2}{(x_2^2 + 1)^2} \right]^T$$

Suppose we begin with $x_1^{(1)} > 1, x_2^{(1)} > 1$, then $\nabla F(x_1^{(1)}, x_2^{(2)})$ will have both of its components negative, and then the next point $(x_1^{(2)}, x_2^{(2)})$ will be further away from the solution comparative to point $(x_1^{(1)}, x_2^{(1)})$.

In this case both of the global and local mapping conditions are satisfied by the transformation but we have no solution of the transformed problem. To avoid this problem, we give some conditions that a transformation should satisfy, which we call the complete mapping conditions

(a) The functions T_j should satisfy the global and local mapping conditions.

(b) If $\lim\limits_{x_j \to \pm\infty} T_j(x_j) = K$, then

If $j = 1, 2, \ldots, m$ then $K = 0$

If $j = m + 1, \ldots, n$ then $K = l_j$ or $K = u_j$

For Example 3, we find that an appropriate transformation is $y_j = T_j(x_j) = \sin x_j$, $j = 1,2$

When we use the periodic transformations, we must take certain precautions. A particular interval in the domain of the function should be designated as the basic part of the function, and any values generated which fall outside of this interval should be displaced back into it.

We give below some appropriate transformations to some specific constraints.

(1) If we have the constraint $y_i \geq 0$, then any one of the following transformations can be used

(a) $y_i = x_i^2$

(b) $y_i = e^{x_i}$

(c) $y_i = |x_i|$

(2) If we have the constraint $0 \leq y_i \leq 1$, then any one of the following transfor-

mations can be used

$$y_i = \sin^2 x_i$$

$$y_i = \frac{e^{x_i}}{e^{x_i} + e^{-x_i}}$$

(3) If we have the constraint $l_i \leq y_i \leq u_i$, then the following transformation can be used

$$y_i = l_i + (u_i - l_i) \sin^2 x_i$$

(4) If we have the constraint $-1 \leq y_i \leq 1$, then the following transformation can be used

$$y_i = \sin x_i$$

(5) If we have the constraint $y_1^2 + y_2^2 \leq 1$, then the following transformation can be used

$$y_1 = \sin x_1 \sin^2 x_2 , y_2 = \cos x_1 \sin^2 x_2$$

(6) If we have the constraint $0 \leq y_i \leq y_j \leq y_k$, then the following transformation can be used

$$y_i = x_i^2 , y_j = x_i^2 + x_j^2 , \quad y_k = x_i^2 + x_j^2 + x_k^2$$

3. Discussion of Convexity

Convexity of a problem is an important feature in the solution of the problem. Here we assume that the original objective function f is convex and we have to determine the convexity of the transformed objective function F. Unfortunately, the following examples illustrate that although the original objective function f is convex, the transformed objective function is not necessarily convex.

Example 4

Minimize $\quad f(y_1 , y_2) = (y_1 + 3)^2 + (y_2 - 4)^2$

Subject to $\quad y_1 , y_2 \geq 0$

The solution is $\quad (0,4)^T$ and $f(y^*) = 9$

Let $y_j = T_j(x_j) = x_j^2 \geq 0 \; , \; j = 1,2$

Then the transformed problem is

$$F(x_1 , x_2) = (x_1^2 + 3)^2 + (x_2^2 - 4)^2$$

$$\nabla^2 f(y_1 , y_2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$\nabla^2 F(x_1 , x_2) = \begin{bmatrix} 12x_1^2 + 12 & 0 \\ 0 & 12x_2^2 - 16 \end{bmatrix}$$

Here we see that although the original objective function f is strictly convex for all values of y, the transformed objective function F is strictly convex only in a neighbourhood of the solution.

Example 5

Minimize $f(y_1, y_2) = 4y_1^2 - 3y_1 y_2 + 4y_2^2$

Subject to $y_1, y_2 \geqslant 0$

The solution is $y^* = (0, 0)^T$ and $f(y^*) = 0$

Let $y_j = T_j(x_j) = x_j^2 \geqslant 0 \;,\; j = 1, 2$

Then the transformed problem is

$$F(x_1, x_2) = 4x_1^4 - 3x_1^2 x_2^2 + 4x_2^4$$

$$\nabla^2 f(y_1, y_2) = \begin{bmatrix} 8 & -3 \\ -3 & 8 \end{bmatrix}$$

and

$$\nabla^2 F(x_1, x_2) = \begin{bmatrix} 48x_1^2 - 6x_2^2 & -12x_1 x_2 \\ -12x_1 x_2 & 48x_2^2 - 6x_1^2 \end{bmatrix}$$

The eigenvalues of $\nabla^2 f(y_1, y_2)$ are 5 and 11 and we see that although the original objective function f is strictly convex for all values of y, the transformed objective function F is not convex.

We can see that every neighbourhood of the solution (0, 0) contains a point of the form $(\delta, 0)$ for which $\nabla^2 F(\delta, 0) = \begin{bmatrix} 48\delta^2 & 0 \\ 0 & -6\delta^2 \end{bmatrix}$ then it is easy to see that the transformed objective function F is not convex in any neighbourhood of the solution.

From Examples 4 and 5 we come to conclude that although the original objective functions are globally convex is both the examples, the transformed objective function F is convex in a neighbourhood of the solution in Example 4 and it is not convex in any neighbourhood of the solution in Example 5.

Now, we give a distinction between the transformed problems, one type is convex in a neighbourhood of the solution and the other is not convex in any neighbourhood of the solution.

We do this by considering two types of solutions of our problems.

Suppose Example 4 illustrates a solution of the first kind, then

(1) $y = T(x)$ is a solution of the first kind if when y_j lies on a boundary of the constraints of the original problem (1) then $f_i'(y) \neq 0$.

(2) $y = T(x)$ is a solution of the second kind if when y_j lies on a boundary of the constraints of the original problem (1) then $f_i'(y) = 0$ for at least one value of i.

Applications

The areas of application of such problems are quite many, particularly in industrial technology. To mention a few are agriculture, transportation, storage problems, ...etc. Also these problems arise in design of experiments, circuit design and least squares data fitting in cases where the variables are subject to bounds. The use of this technique has been found suitable to most of them.

Acknowledgement

The authors are thankful to the learned referees for their fruitful suggestions.

References

[1] **Box, M.J.**, A comparison of several current optimization methods and the use of transformations in constrained problems, *The Computer Journal*, **9:** 67-77 (1966).

[2] **Powell, M.J.D.**, Problems related to unconstrained optimization, *in:* **Murray, W.** (ed.), *Numerical Methods for Unconstrained Optimization*, New York, Academic Press (1972).

[3] **Fletcher, R.**, A new approach to variable metric algorithms, *The Computer Journal*, **13:** 317-322 (1970).

طريقـة للتخلص من القيـود في نماذج الـبرمجـة غـير الخطيـة

أبها جويل و جي . سي . شارما
قسم الرياضيات ، معهد العلوم الإنسانية ، جامعة أجـرا - الهنـد .

المستخلص . تقوم هذه الورقة بدراسة نموذج من نماذج البرمجة غير الخطية ، عندما تكون دالة الأهداف ومشتقاتها مستمرة ، وبوجود بعض القيود العليا أو الصغرى على المتغيرات ، حيث يقوم البحث بإجراء بعض التحويلات للمتغيرات والتي تُحوِّل النموذج إلى نموذج لإيجاد الحلول المثلي لنموذج خالٍ من القيود . كما يقدم البحث الشروط اللازم توافرها في الحل الأدنى لتحقيق كونتكر من الدرجة الأولى ، بالإضافة إلى الشروط اللازم توافرها لكى تحافظ دالة الأهداف المحوَّلة على خاصية التحدب الموجودة في الدالة الأصلية للأهداف .